

NEIMARK-SACKER BIFURCATION IN DELAYED LOGISTIC MAP

HEMANTA K R. SARMAH¹, MRIDUL CHANDRA DAS² & TAPAN K R. BAISHYA³

^{1,2}Department of Mathematics, Gauhati University, Assam, India

³Department of Mathematics, Debraj Roy College, Golaghat, Assam, India

ABSTRACT

The purpose of this paper is to investigate the Neimark-Sacker bifurcation in delayed logistic map. In Neimark-Sacker bifurcation, the fixed point gets converted to a unit circle. In case of our considered map, we observed mode locked state near Neimark-Sacker bifurcation point. Using the analytical technique of Normal form, we have determined the mode locked state in the vicinity of the NS bifurcation point. We have also used numerical tools like final state diagram, Lyapunov exponent, phase portrait and time series plot to establish the transition from periodic to quasiperiodic through mode locked state and finally to chaotic state.

KEYWORDS: Bifurcation, Neimark, Sacker Bifurcation, Normal Form, Quasiperiodic and Mode, Locked States, Limit Cycle

1. INTRODUCTION

The delayed Logistic map $F(x_{n+1}, y_{n+1}) = (\mu x_n(1 - y_n), x_n)$ (1)

Where μ is intrinsic growth rate, is one of the simplest population models in nonlinear dynamical system. This map was first proposed by Maynard Smith [33]. Several authors (Aronson et al [2], Guckenheimer [9], Hale and Kocak [10], Kuznetsov [16], Pounder and Rogers [24], J. C. Sprott [34]) investigated the complex dynamics showed by this map.

Any qualitative change of dynamical behaviour of a system due to variation of a parameter is called bifurcation. For high dimensional discrete time maps, the most probable route to chaos from a fixed point is via at least one Neimark-Sacker (NS) bifurcation followed by persistent zero Lyapunov exponent signifying quasi periodic state and finally a bifurcation into chaos [1]. Orbits that are not periodic and have the zero Lyapunov exponents are said to be quasiperiodic [5, 7, 8, 12, 28, 36]. The NS bifurcation occurs for a discrete system depending on parameter, with a fixed point whose Jacobian has a pair of complex conjugate eigenvalues which cross the unit circle transversally. The NS bifurcation for map is equivalent to the Hopf bifurcation for differential equation [26, 27, 35, 41]. In the case of a supercritical NS bifurcation, a stable focus loses its stability as a parameter is varied with the consequent birth of a stable cycle or quasi-cycle which is known as closed invariant curve. In the case of a subcritical NS bifurcation, a stable focus enclosed by an unstable closed curve loses its stability with the consequent disappearance of the closed invariant curve as a parameter is varied.

In 1971, Ruelle and Takens [30] first proposed the quasi-periodic scenario. It is observed that in both the cases of NS bifurcation for maps and Hopf bifurcation for flows quasi periodic scenario come into picture. Hopf bifurcation in fact is related to the birth and death of limit cycles in a system. The existence of the limit cycles can be observed in fluid dynamics where vortex structures appear [19, 32]. The theory underlying the quasiperiodic route to chaos tells us only that this scenario may lead to chaotic behaviour. In 1978, Newhouse, Ruelle, and Takens [23] proved more rigorously in case of

flows that if the state space trajectories of a system are confined to a three dimensional torus then even a small perturbation of the motion due to external noise, for example, will “destroy” the motion on the torus and lead to chaos and a strange attractor.

The behaviour of orbits near a NS bifurcation [39] reveals many interesting features about the motion of particles in complex systems. Fixed-point solutions are transformed into quasiperiodic states or limit cycles after NS bifurcation. In other words, particles starting in a steady state (or even a mode-locked or synchronized state) end up moving in cycles around one or more centres. Such transformation are observed, for example, when vortex structures appear in fluid dynamics [19,32], in the solutions of multi-agent models of biological swarming [18], in the nonlinear beam oscillations excited by lateral force in sound and vibration physics [3], in the pattern formation and oscillations in a system of self-regulating cells in neural science [29], the collapse of predator populations in biology [22] and in monetary economics [4].

Hutchinson [1948] [13] appears to be the first ecologist to investigate the role of explicit delays in ecological models. He considered the delay differential logistic equation $\frac{dx(t)}{dt} = x(t)(a - b x(t - T))$ with time delay T. Here, it is assumed that the amount of resources available at time t will depend on the density of the species at an earlier time by a delay of T which is in contrasts with the usual logistic differential equation. In this paper, we will examine a discrete analogue of Huthinson's equation that was introduced by Maynard Smith [33] which seems realistic for the following reason:

It is possible that the reproductive rate μ may depend not only on the population density at the time, but on the population density in the past. For example, the reproduction of an herbivorous species will depend on the vegetation, which may in turn depend on how much of the vegetation was eaten by herbivores in the previous year. To gain some idea of the effect of such a delay in the effects of population density on its own increase, a much over-simplified example will be considered. It will be assumed that μ depends only on the population density in the previous year, and neither on the immediate density nor on the density in earlier years.

In modelling seasonally breeding populations whose generations do not overlap, it suffices to keep track of the population once every generation. In such a situation, one can describe the change in the population with a difference equation of the form

$$x_{n+1} = \mu x_n.$$

Where x_n is the population size at the n-th generation and μ is the reproductive rate. Now, for the simplified case mentioned above, we assumes the form

$$\mu = a(1 - x_{n-1})$$

for reproductive rate to arrive at the difference equation

$$x_{n+1} = \mu x_n (1 - x_{n-1}).$$

This equation is almost like the famous logistic map except that the factor regulating the population growth contains a time delay of one generation. To follow the fate of a population, the density of the first two generations must be known. Here the parameter μ is called the intrinsic growth rate.

Now, if we introduce $y_n = x_{n-1}$

Then the above difference equation can be equivalently written as

$$x_{n+1} = \mu x_n(1 - y_n)$$

$$y_{n+1} = x_n \quad (2)$$

2. NEIMARK-SACKER (NS) BIFURCATION

Let F be a two dimensional map from $R^2 \rightarrow R^2$. The NS bifurcation occurs if the Jacobian matrix J of the linearised system of the map F have a complex pair of eigenvalues $\lambda_{1,2}$ so that the following conditions are satisfied [21, 9, 31]

- $|\lambda_{1,2}(a_{NS})| = 1$, but (3)

- $\lambda_{1,2}^j(a_{NS}) \neq 1$ for $j = 1, 2, 3, 4$, (4)

- $\frac{d}{da}(|\lambda_{1,2}(a_{NS})|) = d > 0$ (5)

Where a_{NS} is the bifurcation parameter calculated at the bifurcation point, and d is a constant.

The conditions for NS bifurcation were first derived independently by Neimark in 1959 [21], Sacker in 1964 [31] and by Ruelle and Takens in 1971 [30]. Land ford in 1973 [17] includes the condition $\lambda^5 \neq 1$. A modification to deal with this case may be found in Iooss [15] who gives more precise details of the differentiability conditions required and the regularity of the bifurcating circles.

3. DYNAMICS OF THE MAP

Dynamics of a map gives us the glimpse of its fixed points, periodic attractors and so on. Below we have discussed some of the salient features of the delayed logistic map.

The fixed points of the Delayed Logistic map are given by $F(x, y) = (x, y)$

i.e., $\mu x(1 - y) = x$ and $y = x$

The fixed points are at $(0, 0)$ and $(1 - \frac{1}{\mu}, 1 - \frac{1}{\mu})$

As the fixed point $(0, 0)$ is parameter independent we can conclude that it exists for all values of the parameter.

For $\mu > 0$, the non-trivial fixed point would be $(1 - \frac{1}{\mu}, 1 - \frac{1}{\mu})$.

Now, the Jacobian of the map is, $J = \begin{bmatrix} \mu(1 - y) & -\mu x \\ 1 & 0 \end{bmatrix}$

The Jacobian matrix calculated at the first fixed point $(0, 0)$ is $\begin{bmatrix} \mu & 0 \\ 1 & 0 \end{bmatrix}$ and which has the eigenvalues 0 and μ .

So, from the criterion of stability of a fixed point we conclude that the origin is asymptotically stable if $0 < \mu < 1$ and unstable if $\mu > 1$.

Also, the Jacobian matrix calculated at the other fixed point $(1 - \frac{1}{\mu}, 1 - \frac{1}{\mu})$ is $\begin{bmatrix} 1 & 1 - \mu \\ 1 & 0 \end{bmatrix}$ which has the eigenvalues $\lambda_{1,2} = \frac{1}{2} \pm \sqrt{\frac{5}{4} - \mu}$

So, the fixed point $\left(1 - \frac{1}{\mu}, 1 - \frac{1}{\mu}\right)$ is asymptotically stable if $1 < \mu \leq \frac{5}{4}$ and the eigenvalues are complex conjugate if $\mu > \frac{5}{4}$ and for discussion about the map for those values of μ , we write

$$\lambda_{1,2} = \frac{1}{2} \pm \sqrt{\frac{5}{4} - \mu} = \frac{1}{2} \pm i \sqrt{\mu - \frac{5}{4}} \quad (6)$$

In polar form these can be written as $\lambda_{1,2} = \sqrt{\mu - 1} e^{\pm ic}$ where $c = \arctan \sqrt{4\mu - 5}$

$$\therefore |\lambda_{1,2}| = \sqrt{\mu - 1} \text{ and } \frac{d}{d\mu} |\lambda_{1,2}| = \frac{1}{2\sqrt{\mu-1}}$$

The N S bifurcation occurs under the condition $|\lambda_1| = |\lambda_2| = 1$ leads to $\mu = 2$.

$$\text{and } \frac{d}{d\mu} |\lambda_{1,2}| = \frac{1}{2} = d \neq 0 \quad (7)$$

showing that the NS bifurcation occurs in case of our considered map at $\mu = 2$.

4. NORMAL FORM

The normal form of a bifurcation is a simplified system of equations that approximate the dynamical system in the vicinity of a bifurcation point. Application of the method of normal form for investigation of different complex systems can be found in [6, 25, 26, 38]. It describes the local property of complex systems near bifurcation points. Any bifurcation has the same normal form for all physical models though the coefficients of the normal form change from model to model. In NS bifurcation, it can be achieved by using the method described as follows:

Normal forms are derived on the central manifold [9,39]. To translate the coordinates (x, y) of the dynamical system (1) to the central manifold coordinates (u, v) , we use the following transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} x - x_1 \\ y - y_1 \end{pmatrix} \quad (8)$$

Where (x_1, y_1) is the fixed point and A is a 2×2 transformation matrix. The column of A are the eigenvectors associated to the eigenvalues λ , calculated from the Jacobian matrix of the original system. In fact, the above transformation translates the bifurcating equilibrium point to the origin and brings the linear part into the normal form.

Substituting $\lambda = e^{ic}$ and $\bar{\lambda} = e^{-ic}$ the linear and nonlinear part of the dynamical system can be written in the form

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} \cos(c) & -\sin(c) \\ \sin(c) & \cos(c) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u, v) \\ g(u, v) \end{pmatrix}, \quad (9)$$

$$\text{where the constant } c = \arctan \frac{\text{Im}[\lambda]}{\text{Re}[\lambda]}, \quad (10)$$

Is calculated at the bifurcation point. The first and the second term in the right hand side of the equation (9) gives the linear and the nonlinear parts respectively.

Using the transformation $z = u + iv$ in the complex plane, it is suitable to write equation (9) as

$$z \mapsto \lambda z + f(z \bar{z}) + ig(z \bar{z}). \quad (11)$$

This expression is exactly the dynamical system written in the complex plane and with the bifurcation point located at the origin.

Expanding $f(z \bar{z}) + ig(z \bar{z})$ in a Taylor expansion in z and \bar{z} , we get

$$z \mapsto \lambda z + \frac{1}{2}\xi_{20}z^2 + \xi_{11}z\bar{z} + \frac{1}{2}\xi_{02}\bar{z}^2 + \frac{1}{2}\xi_{02}z^2\bar{z} + \dots \quad (12)$$

where,

$$\xi_{20} = \frac{1}{8}[(f_{uu} - f_{vv} + 2g_{uv}) + i(g_{uu} - g_{vv} - 2f_{uv})] \quad (13)$$

$$\xi_{11} = \frac{1}{4}[(f_{uu} + f_{vv}) + i(g_{uu} + g_{vv})] \quad (14)$$

$$\xi_{02} = \frac{1}{8}[(f_{uu} - f_{vv} - 2g_{uv}) + i(g_{uu} - g_{vv} + 2f_{uv})] \quad (15)$$

$$\xi_{21} = \frac{1}{16}[(f_{uuu} + f_{uvv} + g_{uuv} + g_{vvv}) + i(g_{uuu} - g_{uvv} - f_{uuv} - f_{vvv})] \quad (16)$$

Now considering the trans formation $\eta = \eta(z) = z + O(|z|^2)$, the equation (12) is brought to its simplest form which is called the normal form in the complex plane:

$$\eta \mapsto \lambda\eta + l\eta^2\bar{\eta} + O(|\eta|^5) \quad (17)$$

Where l being a complex number determined by

$$l = \left[\frac{(1-2\lambda)\bar{\lambda}^2}{\lambda-1} \xi_{11}\xi_{20} - \frac{|\xi_{11}|^2}{(1-\lambda^3)} - \frac{2|\xi_{02}|^2}{(1-\lambda)} + \bar{\lambda}\xi_{21} \right] \quad (18)$$

Substituting $\lambda = e^{i\theta}$ and $\eta = re^{i\phi}$, in the r.h.s of the equation (17) we get

$$r(1 + \text{Re}[l]r^2)e^{i(\theta+\phi+\text{Im}[l]r^2)} \quad (19)$$

The radial and angular part of normal form given by (19) can be written as

$$r_{t+1} = r_t[1 + d(a - a_{NS}) + er_t^2] \quad (20)$$

$$\theta_{t+1} = \theta_t + c + sr_t^2 \quad (21)$$

Where c, d e and s are the coefficients of the normal form. The coefficient d is determined using equation (5). Comparing (20) and (21) with (19), we get

$$e = \text{Re}[l] \quad \text{and} \quad s = \text{Im}[l] \quad (22)$$

The properties and stability of solutions near the NS bifurcation depends on the constants c, d, e and s. The stability of a limit cycle is determined by e. If $e < 0$, then the limit cycle is stable where as if $e > 0$ the limit cycle is unstable. In the first case the bifurcation is called supercritical and in the second case the bifurcation is called subcritical.

A detailed study of normal form is found in [9, 14, 15, 16, 20, 37, 38, 39].

5. FINDING ‘E’ AND ‘S’ FOR OUR MAP

The complex conjugates eigenvalues for the delayed logistic map are given by

$$\lambda = \frac{1}{2} + i\sqrt{\mu - \frac{5}{4}} \quad (23)$$

$$\bar{\lambda} = \frac{1}{2} - i\sqrt{\mu - \frac{5}{4}} \quad (24)$$

The eigenvectors corresponding to eigenvalues λ and $\bar{\lambda}$ (these are calculated on the NS line $\mu = 2$) are, respectively given by $\begin{pmatrix} \frac{1}{2} + \frac{\sqrt{3}}{2} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{2} - \frac{\sqrt{3}}{2} \\ 1 \end{pmatrix}$.

To separate the linear part from the nonlinear part, as in equation (6), a new basis is introduced which allows the transformation

$$(x', y') = (x - x_1, y - y_1), \text{ where } x_1 = y_1 = \frac{1}{2} \text{ for } \mu = 2$$

Using the transformation matrix whose columns are the eigenvectors:

$$\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}, \text{ where } A = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \quad (25)$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = A^{-1} \begin{pmatrix} x \\ y \end{pmatrix}, \text{ where } A^{-1} = \begin{pmatrix} \frac{2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 0 & 1 \end{pmatrix} \quad (26)$$

$$\text{Under the transformation } x = \frac{\sqrt{3}}{2} u + \frac{v}{2}$$

$$y = v$$

The system (2) transform into the normal form:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u, v) \\ g(u, v) \end{pmatrix}$$

$$\text{Where } f(u, v) = -(2uv + \frac{2}{\sqrt{3}}v^2) \text{ and } g(u, v) = 0$$

With these function, the coefficients 'e' and 's' can be determined from (22)

$$e = \frac{-4\mu^4 - 4(8+q\sqrt{3}) + 6\mu(8+q\sqrt{3}) + \mu^3(17+q\sqrt{3}) - \mu^2(43+3\sqrt{3}q)}{12(\mu-1)^2(\mu^2-2\mu+4)} \quad (27)$$

$$s = -\frac{4\sqrt{3}\mu^4 + \mu^3(-17\sqrt{3}+q) + \mu^2(39\sqrt{3}+q) - 2\mu(23\sqrt{3}+2q) + 4(5\sqrt{3}+4q)}{12(\mu-1)^2(\mu^2-2\mu+4)} \quad (28)$$

$$\text{Where } q = \sqrt{-5 + 4\mu}$$

In fact, the condition $e \neq 0$, guarantees the existence of an invariant limit cycle above a_{NS} [8, 10] which we have verified in our numerical simulations shown in the following figures [Figure 1].

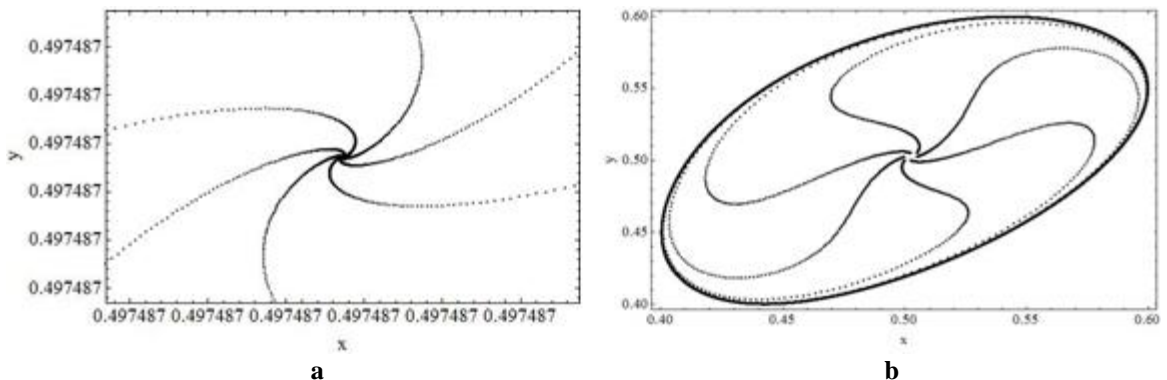


Figure 1: (a) Phase Portrait in Delayed Logistic Map for $\mu = 1.99$, before the NS Bifurcation (b) Stable Invariant Curve in the Delayed Logistic Equation for $\mu = 2.01$, after the NS Bifurcation with $(x_0, y_0) = (0.5, 0.5)$

We have already mentioned that the NS bifurcation conditions are satisfied for the parameter value $\mu = 2$. We plotted the graph of (27) for the range $2 < \mu < 2.05$ (shown in Figure 2) in the vicinity of the NS bifurcation point and found that the coefficient e is always negative for the above mentioned range. So, the limit cycles are stable for the parameter range $2 < \mu < 2.05$.

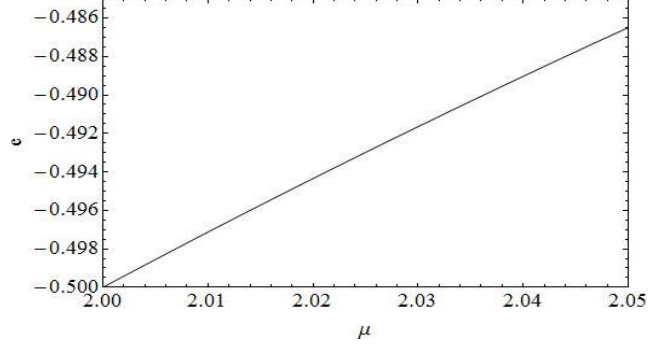


Figure 2: Coefficient e vs. Parameter μ Plot

The Local dynamics described by the normal forms (20) and (21) is very rich. It is possible, for example, to obtain information related to the radius of the limit cycle and the rotation number which is essential for determining whether the state is mode locked or quasiperiodic. Making $r_{t+1} = r_t = r$ in Eq. (20), the radius of the limit cycle is determined by

$$r = \sqrt{-\frac{d}{e}(\mu - \mu_{NS})} \quad (29)$$

Observe that r is always real because the limit cycles exist only for $\mu > \mu_{NS}$, $d > 0$ and $e < 0$.

Therefore the radius of the invariant circle [11, 32] grows as $(\mu - \mu_{NS})^{1/2}$.

Substituting r from Eq. (29) in Eq. (21), the rotation number on the limit cycle can be obtained from

$$\theta_{t+1} = \theta_t + \alpha(\mu), \quad (30)$$

Where

$$\alpha(\mu) = c + sr^2 = c - \frac{sd}{e}(\mu - \mu_{NS}) \quad (31)$$

For points on the NS line ($\mu = \mu_{NS}$), the radius r is zero.

So, from (31), we have $\alpha(\mu_{NS}) = c$,

Where c is obtain from Eq. (10)

$$c = \arctan\left(\frac{Im\lambda}{Re\lambda}\right) = \arctan\sqrt{4\mu - 5} \quad \text{with } \mu > \frac{5}{4}$$

$$\text{Thus we have } \alpha(\mu_{NS}) = \arctan\sqrt{4\mu - 5} \quad (32)$$

It can be observed from (30), that for rational values of $\frac{\alpha(\mu_{NS})}{2\pi}$, points on the limit cycle will be repeated whereas for irrational values of $\frac{\alpha(\mu_{NS})}{2\pi}$, points on the limit cycle will never repeat. Therefore, two cases have to be considered, depending on the ratio $\frac{\alpha(\mu_{NS})}{2\pi}$ [12, 16, 39].

- **Mode-Locking:** If $\frac{\alpha(\mu_{NS})}{2\pi} = \frac{m}{n}$ is a rational number, a periodic regime is obtained. In this case a mode-locked or synchronized state is obtained, with n being the period of the orbit and m its multiplicity.

- **Quasiperiodicity:** If $\frac{\alpha(\mu_{NS})}{2\pi}$ is an irrational number, the motion is quasiperiodic.

The parameter value at which the mode-locked states occur can be obtained analytically by choosing a period n and multiplicity m and substituting $\frac{\alpha(\mu_{NS})}{2\pi} = \frac{m}{n}$ in the relation (34), simplification of

$$\text{Which leads to } \mu = \sec^2\left(\frac{2\pi m}{n}\right) + 1 \quad (33)$$

Using (33) and (27) we get the period-6 mode-locked state for the parameter value $\mu = 2$. Time series plot of the system also make it clear graphically. In the following figures, we have plotted the time series for the parameter values $\mu = 2$ and $\mu = 2.1$ respectively. Though in the first case, period 6 behaviour is visually clear, we fail to get any pattern in the second case.

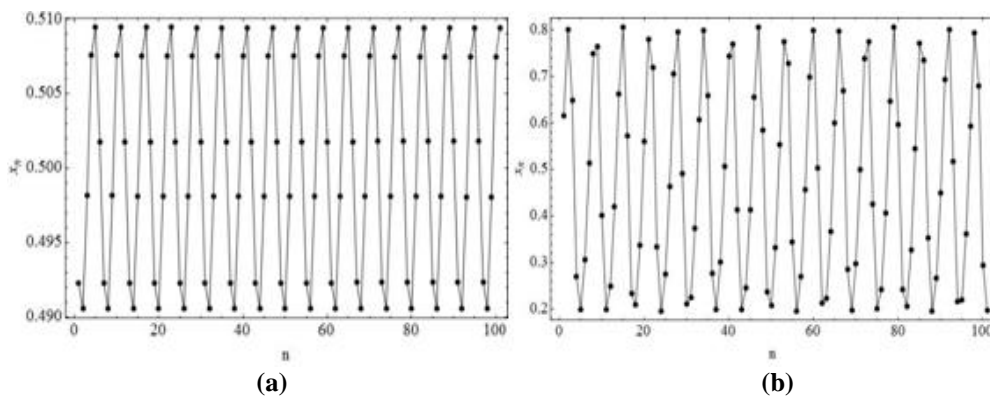


Figure 3: Time Series Plot for the Parameter Value (a) $\mu = 2$ and (b) $\mu = 2.1$

6. NUMERICAL DISCUSSIONS

Bifurcation diagram is primarily a tool to study the long term behaviour of a map which gives all information contained in it with the variation of the control parameter. We study the behaviour of the delayed logistic map at a glance with the help of bifurcation diagram.

Lyapunov exponent is a method of quantifying chaotic behaviour which play very important role in the description of the dynamics of a map. In the bifurcation diagram map like delayed logistic, which exhibits quasiperiodic and chaotic behaviour, it is difficult to distinguish chaos from quasiperiodic state. In such cases, we can draw our conclusion by plotting the Lyapunov exponents. We have used the ‘pull back’ method [40] to find the Lyapunov exponent during our investigation. If the maximum Lyapunov exponent is zero in certain range of the parameter, then we can ascertain quasiperiodic state for that range. Figure 4 shows (a) bifurcation diagram and (b) behaviour of the maximal Lyapunov exponent (h) for different ranges of the parameter where both are shown simultaneously. It shows that up to the parameter value $\mu = 2$, the Lyapunov exponent is negative which indicates periodic behaviour. This fact is supported by the bifurcation diagram of the map. In this range, the bifurcation diagram consists of a single point against each parameter value which is the final state of the map for that particular parameter value. Occurrence of NS bifurcation which we analytically established earlier is supported by zero Lyapunov exponent in our figure. As the parameter is increased further away from $\mu = 2$ where the NS bifurcation occurred, the dynamics of the delayed logistic map is remarkably intricate. By enlarging various parts of the bifurcation diagram we can explore some of this bewildering complexity within it. We have adopted this technique in section 7 for discussion of the 1: 6 mode locked state.

The figure further shows that the maximum Lyapunov exponent for $2 < \mu < 2.175$ is $h \cong 0$. In particular, our calculated value for $\mu = 2.01$ is $h \cong -0.00000037$ and for $\mu = 2.16$ is $h \cong 0.00000624$. So, we can conclude that the

system shows quasiperiodic behaviour within this parameter range. After the parameter crosses the value $\mu = 2.175$ both the Lyapunov exponent become negative up to $\mu = 2.2$ indicating regular behaviour for this range of parameter values. In particular we calculated the Lyapunov exponent for $\mu = 2.19$ and found it to be $h = -0.168024 \dots$. The bifurcation diagram supports this conclusion. Ultimately when the parameter crosses the value $\mu = 2.23$ the maximum Lyapunov exponent is found to be positive indicating chaotic behaviour.

The arrow in Figure 4(b) indicates the range of quasiperiodic motion which begins at the NS bifurcation point where a fixed-point solution is transformed into a quasiperiodic motion. The quasiperiodic motion continues with the increase of the parameter value μ and eventually it either goes to chaos ($h > 0$) or it continues with the existing state until the attractor gets destroyed in a boundary crisis [27, 36]. In the quasiperiodic region the bifurcation diagram is filled out like the chaotic region making it difficult to draw conclusions about the state depending solely on the bifurcation diagram. At this confusing state we must take help of the values of the Lyapunov exponent which is zero for quasiperiodic state and positive for chaotic state. For all the figures the trajectories were initialised at $(x_0, y_0) = (0.1, 0.1)$. The final state diagram even in the case of quasiperiodic state and mode-locked state in the neighbourhood of the NS bifurcation point looks similar. We can distinguish their differences by magnifying the final state diagram in the neighbourhood of the NS bifurcation point. So, we have drawn the figure in 4(c) which is the blown up portion within the square shown in figures 4(a). We have discussed the case mentioned below following the above technique.

Mode-Locked State for $\mu = 2$

From the bifurcation diagram and the Lyapunov exponent which are shown in figure 4(a) and 4(b) we can conclude that the quasiperiodic state continues from $\mu = 2.0$ to $\mu = 2.175$ approximately. After the quasiperiodic state we have noticed a periodic state up to $\mu = 2.2$ and then the attractor goes to chaos before it gets destroyed in a boundary crisis near $\mu = 2.271 \dots$. The measures of the Lyapunov exponents in this case further shows that the system is strongly chaotic. The figure 4(c) which is the magnification of the portion within the square in 4(a) which is in the neighbourhood of the NS bifurcation point shows that it meets with the 1:6 mode-locked state.

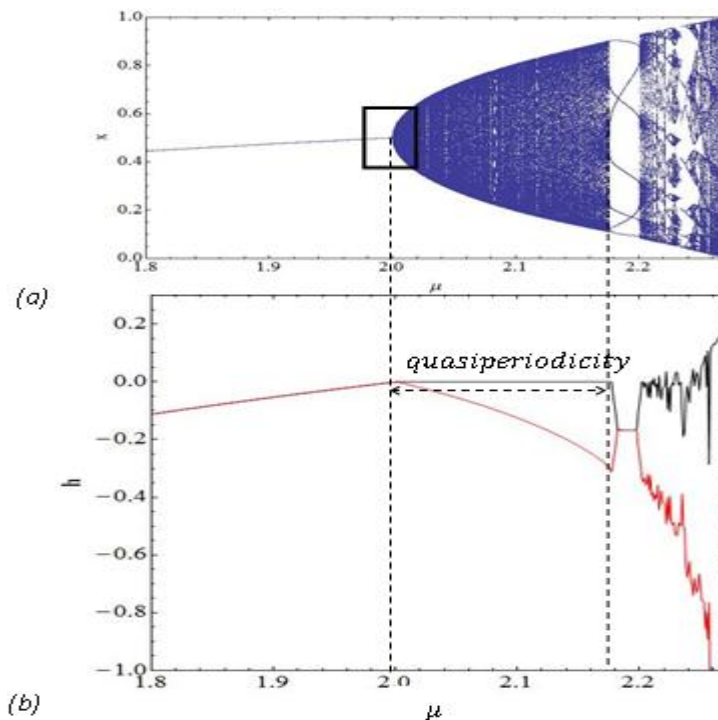


Figure 4: (a) Bifurcation Diagram and (b) Maximal Lyapunov Exponent

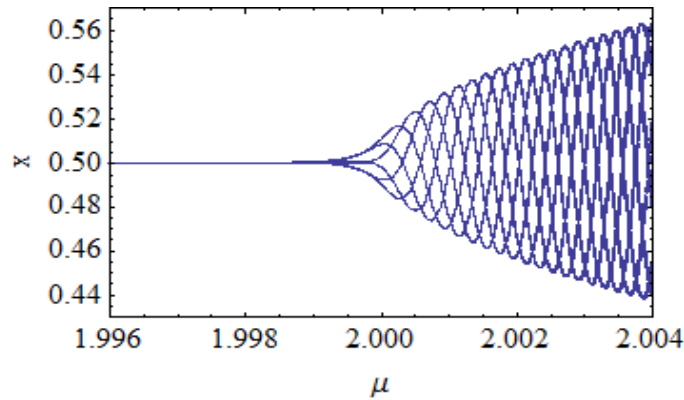


Figure 4: (c) Blown up Portion of the Bifurcation Diagram inside the Square

The following figures 4(d), 4(e) which are the phase portraits just before and after the NS bifurcation point justifies our claim. We have drawn the figures with the initial point ($x = 0.5, y = 0.5$) and allowing the attractor 10000000 number of iterations and then neglecting 9990000 initial iterations. For the figure 4(d) the parameter value μ was taken to be 1.99 which is just prior to the NS bifurcation point and for the figure 4(e) the parameter value μ was taken to be 2.00000001 which is just after the NS bifurcation. The single point in the figure 4(d) ascertains that the system is regular with period 1 and the 6 points in the figure 4(e) shows that the system attains the mode-locked state 1: 6.

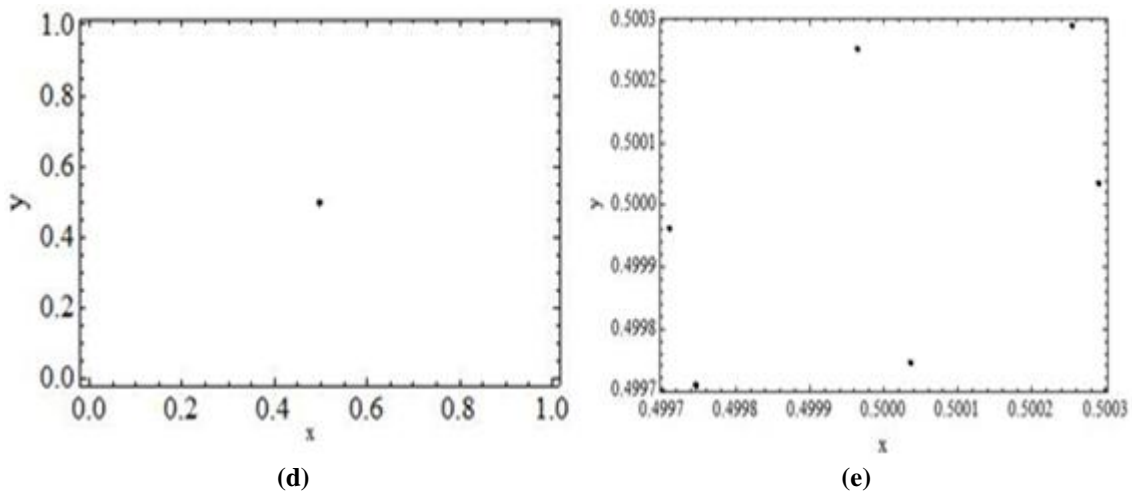


Figure 4: (d) Phase Portrait at $\mu = 1.99$ Just before the NS Bifurcation Point (e) Phase-Portrait at $\mu = 2.00000001$ Just after the NS Bifurcation Point

Evolution of the Attractor at 1: 6 Mode-Locked State

From the earlier discussions it is quite clear that the NS bifurcation is marked by creation of limit cycles which keeps on growing in radius and ultimately gets deformed when the chaotic state is reached. The basic difference which takes place in case of mode-locked state and the quasiperiodic state in the immediate vicinity of the NS bifurcation point is that in the earlier case the state of the system changes from period one to two or more (depending upon the mode-locked state) and then it forms a stable limit cycle whereas in the quasiperiodic case the limit cycle gets immediately created just after crossing the NS bifurcation point. In the following diagrams we have shown the evolution of the limit cycles which is created after the NS bifurcation point in case of 1: 6 mode-locked state. The implication of the first two figures for $\mu = 1.99$ and $\mu = 2.00001$ were already made clear in describing the figures 4(d) and 4(e). The figures for $\mu = 2.01, 2.1$ shows the existence of limit cycles signifying quasiperiodic state and the last figure for $\mu = 2.23$ shows the deformation of the limit cycle which shows that chaos creeps in to the system.

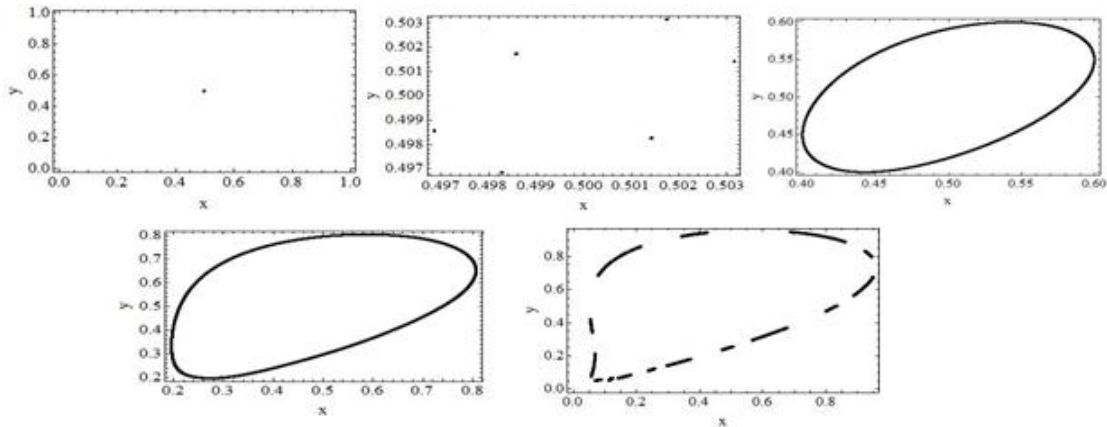


Figure 5: Phase Portrait for the Parameter Value $\mu = 1.99, 2.00001, 2.01, 2.1, 2.23$

In fact, the limit cycles which come in to the picture grows larger with increasing radius r (shown in figure 6(a) and 6(b)) with the increase in the parameter μ , until they get deformed when the chaotic state is reached.

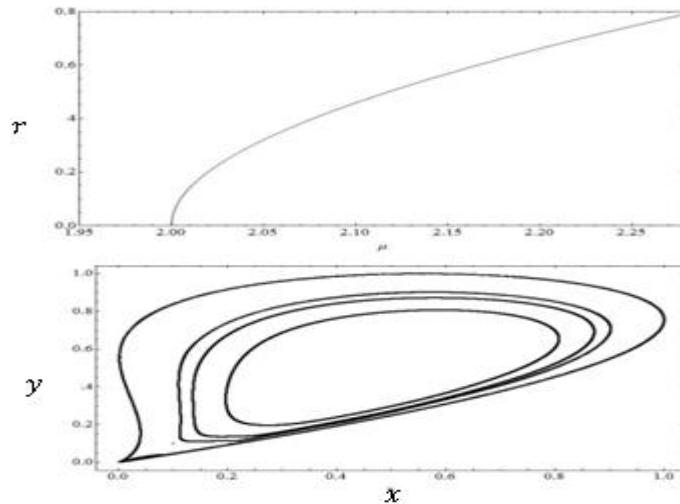


Figure 6: (a) Radius of the Limit Cycle Obtained Analytically is Plotted with Respect to μ . (b) Four Trajectories in the Phase Space With $\mu = 2.1, 2.15, 2.175, 2.27$

Below, we have compared the analytical radius obtained from Eq. (29) of the limit cycles with numerical radius. The numerical radius of a limit cycle is an average radius over all points of trajectory. Figure 7 shows a comparison between the radius of the limit cycle calculated from Eq. (29) (black line) with the radius obtained by numerical simulation (blue line).

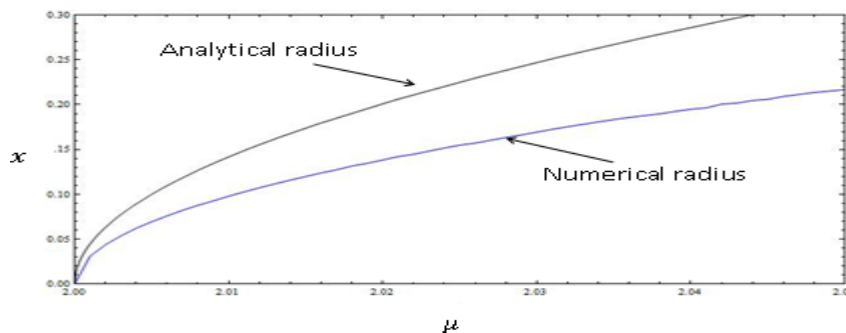


Figure 7: Radius of the Limit Cycle Obtained Analytically (Black Line) and Numerically (Blue Line)

Both curves are plotted as a function of the bifurcation parameter μ . The agreement between numerical and

analytical result decreases very fast. The reason of this disagreement is that - far away from the NS bifurcation point, when the chaotic region is approached, limit cycles get totally deformed and the numerical calculation of the limit cycle radius gets worse.

7. OBTAINING 'e' FOR $\mu = 2$

From the nonlinear terms $f(u, v) = -(2uv + \frac{2}{\sqrt{3}}v^2)$ and $g(u, v) = 0$ we have the following

$$f_{uu} = 0, f_{uv} = -2, f_{vv} = -\frac{4}{\sqrt{3}}$$

$$g_{uu} = 0, g_{uv} = 0, g_{vv} = 0$$

$$\text{Thus, we get } \xi_{20} = \frac{1}{8}[(f_{uu} - f_{vv} + 2g_{uv}) + i(g_{uu} - g_{vv} - 2f_{uv})] = \frac{\sqrt{3}}{6} + \frac{i}{2}$$

$$\xi_{11} = \frac{1}{4}[(f_{uu} + f_{vv}) + i(g_{uu} + g_{vv})] = -\frac{1}{\sqrt{3}}$$

$$\xi_{02} = \frac{1}{8}[(f_{uu} - f_{vv} - 2g_{uv}) + i(g_{uu} - g_{vv} + 2f_{uv})] = \frac{\sqrt{3}}{6} - \frac{i}{2}$$

$$\text{Now, for } \mu = 2 \text{ we get, } \lambda = \frac{1+i\sqrt{3}}{2}, \bar{\lambda} = \frac{1-i\sqrt{3}}{2}$$

Substituting the value of $\xi_{20}, \xi_{11}, \xi_{02}, \lambda, \bar{\lambda}$ in (22) which is given by

$$e = \text{Re}\left[\frac{(1-2\lambda)\bar{\lambda}^2}{\lambda-1}\xi_{11}\xi_{20} - \frac{|\xi_{11}|^2}{(1-\lambda^3)} - \frac{2|\xi_{02}|^2}{(1-\lambda)} + \bar{\lambda}\xi_{21}\right]$$

$$\text{We get } e = -\frac{1}{2} < 0$$

Thus, the bifurcation is supercritical implying existence of an attracting invariant closed curve surrounding $(\frac{1}{2}, \frac{1}{2})$ for $\mu > 2$.

The above fact can be verified with the help of phase portraits of the delayed logistic map for $\mu = 1.99, 2.0, 2.01$ and 2.1 which we have shown below. When $\mu < 2$, the origin is asymptotically stable and all iterates starting within its domain of attraction spiral in to it. When $\mu > 2$, the origin is unstable and all iterates either spiral in or spiral out to a smooth closed invariant curve enclosing the origin depending on the choice of the initial point.

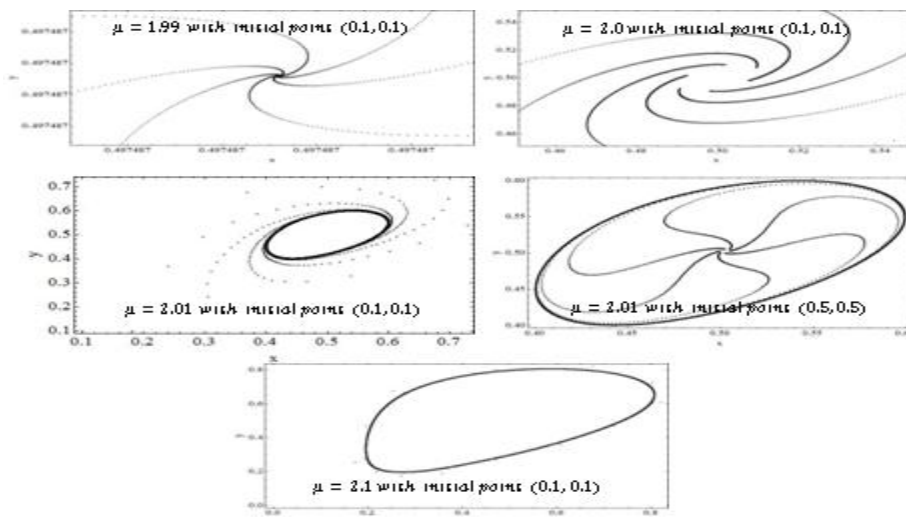


Figure 8: Phase Portrait in Delayed Logistic Map for $\mu = 1.99, 2.0, 2.01, 2.1$

8. CONCLUSIONS

The technique of normal form gives good results near the NS bifurcation point. Using normal form we can determine whether the motion is quasiperiodic or mode-locked and in this paper we have given a numerical simulation for the delayed logistic map which is in conformity with the results obtained from the normal form. After the NS bifurcation, a fixed point solution is transformed into a mode-locked or synchronized state. For very specific conditions, the fixed point solution is transformed into a mode-locked or synchronised state which occurs in the delayed logistic map. Our numerical simulation has shown that the delayed logistic map exhibits a Neimark-Sacker bifurcation where a 1:6 mode-locked state occurs at the parameter value $\mu = 2$.

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